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AI-Driven Dynamic Pricing and Advertising in E-Commerce Competition: A Differential Game Analysis of Coordination Reversal and Welfare

Mian Haseeb Mushtaq[1], Aiman Ejaz[2], Talha Zubair[3], Hasnain Ahmad[4]

[1]Department of Software Engineering, Lahore Garrison University, Pakistan

Email: haseebmuhstaq@gmail.com

[2]COMSATS University Islamabad, Abbottabad Campus, Pakistan

Email: aiman.ejaz009@gmail.com

[3] Independent researcher, Newcastle Upon Tyne, United kingdom

Email: Talha.jaan3@gmail.com

[4] Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan

Email: hasahmad9982@gmail.com

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Corresponding author:
Mian Haseeb Mushtaq
Email:
haseebmuhstaq@gmail.com

ABSTRACT

This paper develops a continuous-time differential game to examine how artificial intelligence shapes strategic pricing and advertising competition among e-commerce sellers. Building on recent reinforcement-learning evidence of counterintuitive lower-price outcomes under consumer search responsiveness, the analysis incorporates AI capability through demand sensitivity and advertising effectiveness within a Nerlove-Arrow (1962) goodwill framework. A closed-form coordination-reversal condition is derived in a static-goodwill benchmark, showing that coordinated equilibrium prices fall below non-cooperative Nash equilibrium prices if and only if the advertising-spillover effect dominates the cross-price substitution effect. The mechanism is then embedded in a continuous-time differential game with endogenous goodwill, and the feedback Nash equilibrium conditions are formulated through a coupled Hamilton-Jacobi-Bellman system. In the static benchmark, higher AI capability is shown to expand the parameter region supporting lower prices under coordination through the advertising-effectiveness channel. Welfare analysis reveals that lower prices do not automatically guarantee higher consumer surplus when advertising and goodwill adjust; the exact demand condition for consumer-surplus improvement is derived. The framework provides an analytical characterization of a mechanism that can generate the lower-price coordination outcome reported in recent computational research, clarifying conditions under which multidimensional coordination can be associated with consumer-beneficial rather than purely anticompetitive outcomes.

Introduction

The digital transformation of retail has increasingly shifted pricing and advertising decisions toward automated, high-frequency decision systems. Artificial intelligence, including reinforcement-learning methods, is increasingly being used to support these decisions in e-commerce markets. E-commerce platforms such as Amazon, Alibaba, and JD.com have become arenas where millions of sellers compete in real time, adjusting prices and advertising bids with unprecedented frequency.

Recent computational research has investigated a striking phenomenon: when AI agents simultaneously optimize pricing and advertising in environments with consumer search responsiveness, they can coordinate on equilibrium outcomes featuring lower prices than non-cooperative benchmarks (Zhao & Berman, 2025). This counterintuitive result emerges because algorithms discover that reducing advertising expenditures is a more effective coordination mechanism than raising prices. However, while these findings have been reported in reinforcement-learning simulations, the theoretical foundations remain underexplored.

This paper addresses this gap by developing a differential game that analytically characterizes the equilibrium of AI-driven pricing and advertising competition. The central research question is: Under what conditions does strategic coordination in simultaneous pricing and advertising produce lower equilibrium prices?

The contributions are threefold. First, a closed-form coordination-reversal condition is derived in a static-goodwill benchmark, providing an analytical characterization of a mechanism that can generate the lower-price coordination outcome identified in recent computational research. Second, AI capability is shown to expand the parameter region supporting lower prices under coordination in the static benchmark. Third, welfare conditions are characterized, showing that lower prices do not automatically guarantee higher consumer surplus when advertising and goodwill adjust.

This paper builds on three scholarly traditions. Foundational work by Dockner and Feichtinger (1986) established differential-game frameworks for dynamic advertising and pricing. Mahdiraji et al. (2021) extended this to concurrent pricing and advertising decisions using Nerlove-Arrow (1962) goodwill dynamics. Schlosser (2017) analyzed stochastic dynamic pricing and advertising in isoelastic oligopoly models. Calvano et al. (2020) demonstrated that Q-learning agents can tacitly collude on supra-competitive prices in single-dimensional price competition. Subsequent work by Deng et al. (2025) showed that collusion depends critically on the learning algorithm and environment. Zhao and Berman (2025) extended this to two-dimensional competition, investigating lower-price outcomes under high search responsiveness.

The contribution differs from Zhao and Berman's in kind: they investigate the phenomenon using multi-agent reinforcement learning calibrated with Amazon data; the present analysis provides an analytical characterization of a mechanism that can generate the lower-price coordination outcome. Rather than claiming discovery of lower-price coordination, analytical conditions are derived that characterize when multidimensional coordination can generate lower prices, thereby providing a theoretical interpretation of the mechanism identified in recent reinforcement-learning research.

Model

Setup

Consider a duopolistic e-commerce market with two competing sellers, indexed by $i \in \{1,2\}$. Each seller makes two strategic decisions at each continuous time $t \in [0, \infty)$:

1. The retail price $p_i(t) \in R+$
2. The advertising intensity $a_i(t) \in R+$

Consumer Demand

Consumer demand for seller i's product follows a Nerlove-Arrow (1962) specification with search responsiveness:

$$D_i = \alpha_0 + \alpha_1 G_i - \beta(q_i)p_i + s\gamma p_j + \lambda(q_i)a_i - s\mu a_j$$

Where:

- $G_i(t)$ is brand goodwill
- $s \in (0,1)$ is a search responsiveness parameter (higher s means greater cross-firm responsiveness)
- $\beta(q_i) > 0$ is price sensitivity, decreasing in AI capability q_i
- $\gamma \in (0, \beta)$ captures cross-price effects
- $\lambda(q_i) > 0$ is advertising effectiveness, increasing in qi
- $\mu > 0$ Captures competitive advertising spillovers.

Demand shocks are abstracted from in the deterministic equilibrium. While AI may improve forecast precision, this channel is discussed as an informational interpretation but is not explicitly modeled.

AI capability: $q_i \in [0,\infty)$ affects price sensitivity reduction through $\beta(q_i) = \beta_0/(1+\kappa_\beta q_i)$ with $\kappa_\beta > 0$, and advertising effectiveness enhancement through $\lambda(q_i) = \lambda_0 + \kappa_\lambda q_i$ with $\kappa_\lambda > 0$.

Goodwill Dynamics

Brand goodwill evolves according to:

$$\dot{G}_i(t) = a_i(t) - \delta G_i(t), \quad G_i(0) = G_{i0}$$

where $\delta > 0$ is the depreciation rate.

Costs and Profits

Each seller faces constant marginal cost c_i and quadratic advertising cost:

$$C(a_i) = \frac{\eta}{2} a_i^2, \qquad \eta > 0$$

Instantaneous profit is:

$$\pi_i(t) = (p_i(t) - c_i)D_i - \left(\frac{\eta}{2}\right) a_i(t)^2$$

Each seller maximizes discounted future profits:

$$V_i(G_i, G_j) = \max_{p_i(.), a_i(.)} \int_0^\infty e^{-rt} \pi_{i(t)}\, dt$$

Where $r > 0$ is the discount rate.

Strategic Regimes

Two strategic regimes are analyzed. Regime N is the non-cooperative Nash equilibrium where each seller solves $\max_{pi,ai} J_i$. Regime C is the joint-profit coordinated benchmark where sellers jointly solve $\max_{p1,p2,a1,a2} (J_1 + J_2)$ the coordinated regime is modeled as a joint-profit cooperative benchmark that provides an analytical upper-level representation of coordinated behavior. It does not model the learning or communication mechanism through which decentralized algorithms attain coordination.

Static-Goodwill Benchmark

Assumptions for Interior Equilibrium

The following assumptions are maintained.

Structural assumptions

(S1) $\alpha_0 > 0$ (Positive baseline demand)

(S2) $\beta > s\gamma$ (cross-price effect does not dominate own-price effect)

(S3) $2\beta - s\gamma - \frac{\lambda(\lambda - s\mu)}{\eta} > 0$ (Nash denominator positivity)

(S4) $2\eta(\beta - s\gamma) > (\lambda - s\mu)^2$ (Cooperative denominator/concavity condition)

(S5) $\lambda > s\mu$ whenever an interior positive coordinated advertising solution is imposed

Feasibility/Interiority

(F1) The relevant stationary solutions satisfy $p_i > 0, a_i > 0,$ and $D_i > 0$.

Concavity conditions

(C1) Individual-firm strict concavity: $2\beta\eta - \lambda^2 > 0$

(C2) Cooperative strict concavity (antisymmetric mode):

$$2\eta\,(\beta + s\gamma) > (\lambda + s\mu)^2$$

Note: (S4) is $2\eta\,(\beta - s\gamma) > (\lambda - s\mu)^2$, which is the symmetric-mode concavity condition. The antisymmetric-mode condition is stated separately as (C2). Under the stated denominator, concavity, and interior-feasibility conditions, the stationary solutions derived below are interior optima.

Equilibrium with No Goodwill Dynamics (α1 = 0, c = 0)

We first analyze the tractable benchmark with no goodwill dynamics. This isolates the core advertising-spillover mechanism and provides the foundation for understanding the mechanism before adding dynamic complexity.

Demand:

$$D_i = \alpha_0 - \beta p_i + s\gamma p_j + \lambda a_i - s\mu a_j$$

Regime N: Nash Equilibrium

First-order conditions

$$D_i - \beta p_i = 0 \quad \Rightarrow \quad D_i = \beta p_i$$

$$\lambda p_i - \eta a_i = 0 \quad \Rightarrow \quad a_i = \frac{\lambda}{\eta} p_i$$

At symmetry ($p_i = p_j = p^N,\ a_i = a_j = a^N$)

$$D^N = a_0 - (\beta - s\gamma) p^N + (\lambda - s\mu) a^N$$

Using $D^N = \beta p^N$:

$$\beta p^N = \alpha_0 - (\beta - s\gamma) p^N + (\lambda - s\mu) \frac{\lambda}{\eta} p^N$$

$$p^N \left[2\beta - s\gamma - \frac{\lambda(\lambda - s\mu)}{\eta} \right] = \alpha_0$$

$$p^N = \frac{\alpha_0}{2\beta - s\gamma - \dfrac{\lambda(\lambda - s\mu)}{\eta}}$$

$$a^N = \frac{\lambda}{\eta} p^N$$

Regime C: Joint-Profit Coordinated Benchmark

Joint first-order conditions

Price (accounting for cross-price effect on rival profit):

$$D_i - \beta p_i + s\gamma p_j = 0$$

At symmetry:

$$D^C - (\beta - s\gamma) p^C = 0 \quad \Rightarrow \quad D^C = (\beta - s\gamma) p^C$$

Advertising (accounting for cross-advertising effect):

$$(\lambda - s\mu) p^C - \eta a^C = 0 \quad \Rightarrow \quad a^C = \frac{\lambda - s\mu}{\eta} p^C$$

Demand at symmetry:

$$D^C = \alpha_0 - (\beta - s\gamma)p^C + (\lambda - s\mu)a^C$$

Using$D^C = (\beta - s\gamma)p^C$:

$$(\beta - s\gamma)p^C = \alpha_0 - (\beta - s\gamma)p^C + (\lambda - s\mu)\frac{(\lambda - s\mu)}{\eta}\ p^C$$

$$p^C\left[2(\beta - s\gamma) - \frac{(\lambda - s\mu)^2}{\eta}\right] = \alpha_0$$

$$p^C = \frac{\alpha_0}{2(\beta - s\gamma) - \frac{(\lambda - s\mu)^2}{\eta}}$$

$a^C = \frac{\lambda - s\mu}{\eta}\ p^C$

Coordination Reversal Theorem

Theorem 1 (Static Coordination Reversal): Under the stated denominator, concavity, and interior-feasibility conditions:

$$p^C < p^N \Leftrightarrow \mu(\lambda - s\mu) > \eta\gamma$$

Proof:

Define:

$$D_N = 2\beta - s\gamma - \frac{\lambda(\lambda - s\mu)}{\eta}$$

$D_C = 2\beta - 2s\gamma - \frac{(\lambda - s\mu)^2}{\eta}$

For $\alpha_0 > 0$ and $D_N, D_C > 0$:

$$p^C < p^N \iff D_C > D_N$$

$D_C - D_N = \left[2\beta - 2s\gamma - \frac{(\lambda - s\mu)^2}{\eta}\right] - \left[2\beta - s\gamma - \frac{\lambda(\lambda - s\mu)}{\eta}\right]$

$$= -s\gamma + \frac{\lambda(\lambda - s\mu) - (\lambda - s\mu)^2}{\eta}$$

$$= -s\gamma + \frac{(\lambda - s\mu)[\lambda - (\lambda - s\mu)]}{\eta}$$

$$= -s\gamma + \frac{s\mu(\lambda - s\mu)}{\eta}$$

$$= s\left[\frac{\mu(\lambda - s\mu)}{\eta} - \gamma\right]$$

For s > 0:

$$D_C > D_N \iff \frac{\mu(\lambda - s\mu)}{\eta} > \gamma$$

$$\iff \mu(\lambda - s\mu) > \eta\gamma$$

Economic Intuition: The reversal occurs when the savings generated by internalizing the negative advertising spillover exceed the price-competition effect created by internalizing cross-price substitution. The advertising-spillover channel ($\mu(\lambda - s\mu)$) captures the benefit from internalizing the negative cross-advertising effect, which can make coordinated advertising lower than decentralized advertising. The cross-price effect ($\eta\gamma$) captures the tendency of coordination to raise prices through internalized cross-price competition.

Corollary 1 (Reversal Region): A nonempty reversal region exists iff $\mu\lambda > \eta\gamma$. The reversal region is:

$$0 < s < \min\left\{1, \frac{\mu\lambda - \eta\gamma}{\mu^2}\right\}$$

Proof: From $\mu(\lambda - s\mu) > \eta\gamma$:

$$\mu\lambda - s\mu^2 > \eta\gamma \Rightarrow s < \frac{\mu\lambda - \eta\gamma}{\mu^2}$$

Since $s \in (0, 1)$ the actual reversal set is $\left(0, \min\left\{1, \frac{\mu\lambda-\eta\gamma}{\mu^2}\right\}\right)$.

Table 1: Static Equilibrium Summary

Variable	Regime N (Nash)	Regime C (Coordinated)
Price	$p^N = \dfrac{\alpha_0}{2\beta - s\gamma - \dfrac{\lambda(\lambda - s\mu)}{\eta}}$	$p^C = \dfrac{\alpha_0}{2(\beta - s\gamma) - \dfrac{(\lambda - s\mu)^2}{\eta}}$
Advertising	$a^N = \dfrac{\lambda}{\eta} p^N$	$a^C = \dfrac{\lambda - s\mu}{\eta} p^C$
Demand	$D^N = \beta p^N$	$D^C = (\beta - s\gamma)p^C$

Dynamic Differential Game Extension

Dynamic State and Control Structure

The state variables are G_1, G_2, the controls are p_i, a_i, and the dynamics are $\dot{G}_i = a_i - \delta G_i$.

Non-cooperative Hamilton-Jacobi-Bellman Conditions

For firm i, with value function $V_i^N(G1, G2)$:

$$rV_i^N = \max_{p_i, a_i} \left\{(p_i - c_i)D_i - \frac{\eta}{2}a_i^2 + V_{Gi}^N(a_i - \delta G_i) + V_{G_j}^N\big(a_j - \delta G_j\big)\right\}$$

The first-order conditions for price are $D_i^N = \beta(q_i)\big(p_i^N - c_i\big)$ and for advertising

$$a_i^N = \frac{\lambda(q_i)\big(p_i^N - c_i\big) + V_{Gi}^N}{\eta}.$$

Cooperative Hamilton-Jacobi-Bellman Conditions

Define a single cooperative value function WC (G1, G2):

$$rW^C = \max_{p_1, p_2, a_1, a_2} \left\{\sum_{i=1}^{2}\left[(p_i - c_i)D_i - \frac{\eta}{2}a_i^2\right] + W_{G_1}^C(a_1 - \delta G_1) + W_{G_2}^C(a_2 - \delta G_2)\right\}$$

The first-order conditions at symmetry are

$$D^C = (\beta - s\gamma)(p^C - c)$$

$$a^C = \frac{(\lambda - s\mu)(p^C - c) + W_G^C}{\eta}$$

Candidate Quadratic Value Functions and Affine Feedback Policies

The investigation explores whether a quadratic value function and affine feedback controls provide a closed-form solution class for the dynamic game.

For Nash:

$$V^N(G_i, G_j) = v_0^N + v_1^N(G_i + G_j) + \frac{1}{2}v_3^N({G_i}^2 + {G_j}^2) + v_4^N G_i G_j$$

$$p_i^*(G_i, G_j) = P_0^N + P_1^N G_i + P_2^N G_j, \quad a_i^*(G_i, G_j) = A_0^N + A_1^N G_i + A_2^N G_j$$

For cooperative:

$$W^C(G_i, G_j) = w_0 + w_1(G_i + G_j) + \frac{1}{2}w_3({G_i}^2 + {G_j}^2) + w_4 G_i G_j$$

$$p_i^*(G_i, G_j) = P_0^C + P_1^C G_i + P_2^C G_j, \quad a_i^*(G_i, G_j) = A_0^C + A_1^C G_i + A_2^C G_j$$

Policy-Coefficient System

Substitution of the candidate policies into the first-order conditions yields a system of algebraic equations for the policy coefficients. This system is presented in Appendix A. If it admits a solution satisfying the relevant concavity and transversality conditions, the affine policies constitute a feedback Nash equilibrium.

Hamilton-Jacobi-Bellman Coefficient Matching

After the policy coefficients are determined, substitution into the HJB equations and matching of coefficients in $1, G_i, G_j, G_i^2, G_i G_j, G_j^2$ yields the HJB coefficient system, also presented in Appendix A.

Symmetric Steady-State Characterization

At steady state:

$$G_i = G_j = \frac{a}{\delta}$$

The shadow values are:

$$V_G^{N,ss} = v_1^N + (v_3^N + v_4^N)\frac{a^N}{\delta}$$

$$W_G^{C,ss} = w_1 + (w_3 + w_4)\frac{a^C}{\delta}$$

The steady-state first-order conditions imply the following effective advertising-to-margin representations:

$$a^N = K_N(p^N - c), \quad K_N = \frac{\lambda}{\eta} + \frac{V_G^{N,ss}}{\eta(p^N - c)}$$

$$a^C = K_c(p^C - c), \quad K_c = \frac{\lambda - s\mu}{\eta} + \frac{W_G^{C,ss}}{\eta(p^C - c)}$$

K_N and K_C are effective steady-state advertising-to-margin ratios. They are definitions and identities derived from the HJB first-order conditions, not solved equilibrium coefficients.

Conditional Dynamic Coordination-Reversal Characterization

Let:

$$A^N = \alpha_0 + \beta c - \Lambda^N K^N c, \quad A^C = \alpha_0 + (\beta - s\gamma)c - \Lambda^C K^C c$$

$$B^N = 2\beta - s\gamma - \Lambda^N K^N, \quad B^C = 2(\beta - s\gamma) - \Lambda^C K^C$$

with $B^N > 0$, $B^C > 0$, and conditional on the existence of the corresponding steady-state feedback equilibria. Then:

$$p^N = \frac{A^N}{B^N}, \quad p^C = \frac{A^C}{B^C}$$

Proposition 1 (Conditional Dynamic Coordination-Reversal Characterization): Conditional on the existence of interior steady-state feedback equilibria with positive denominators:

$$p^C < p^N \iff A^C B^N < A^N B^C$$

Unlike the static benchmark, this condition is not closed-form because K^N and K^C depend on the endogenous shadow values generated by the HJB coefficient system. The static benchmark is recovered as the special case $\alpha 1 = 0$ and $c = 0$, in which goodwill does not enter demand and marginal cost is normalized to zero, yielding:

$$\mu(\lambda - s\mu) > \eta\gamma$$

Admissibility Conditions for Candidate Feedback Equilibrium

A candidate feedback equilibrium is admissible if the HJB coefficient system admits a solution, the value function is strictly concave with $v_3^N < 0, \quad |v_4^N| < -v_3^N, \quad w_3 < 0, \quad |w_4| < -w_3$ the transversality condition $\lim_{T\to\infty} e^{-rT} V\big(G(T)\big) = 0$ holds, controls are non-negative with $p_i^*, a_i^* \geq 0$, and denominators are positive with $B^N, B^c > 0$.

AI Capability and Equilibrium

AI Effect on the Reversal Boundary

Proposition 2 (AI Expands the Static Reversal Boundary): In the static-goodwill benchmark, define the reversal threshold:

$$\tilde{s}(q) = \frac{\mu\lambda(q) - \eta\gamma}{\mu^2}$$

Higher AI capability q expands the reversal region:

$$\frac{d\tilde{s}}{dq} = \frac{\kappa\lambda}{\mu} > 0$$

Proof:

$$\lambda(q) = \lambda_0 + \kappa_\lambda q, \quad \text{so} \quad \frac{d\tilde{s}}{dq} = \frac{\kappa_\lambda \mu}{\mu^2} = \frac{\kappa_\lambda}{\mu} > 0$$

In the static benchmark, AI capability expands the reversal boundary through the advertising-effectiveness channel; the price-sensitivity channel does not enter the reversal threshold because β cancels from the static reversal condition.

AI Effect on Equilibrium Prices

Proposition 3 (AI and Nash Price — Static Benchmark): Under the stated denominator, concavity, and interior-feasibility conditions, and the sufficient sign condition $2\lambda > s\mu$:

$$\frac{\partial p^N}{\partial q} = \frac{\alpha_0}{{D^N}^2}\left[-2\beta_q + \frac{(2\lambda - s\mu)\lambda_q}{\eta}\right] > 0$$

Proof:

$$p^N = \frac{\alpha_0}{D^N} \quad D^N = 2\beta - s\gamma - \frac{\lambda(\lambda - s\mu)}{\eta}$$

$$\frac{\partial p^N}{\partial q} = -\frac{\alpha_0}{{D^N}^2}\frac{\partial D^N}{\partial q}$$

$$\frac{\partial D^N}{\partial q} = 2\beta_q - \frac{(2\lambda - s\mu)\lambda_q}{\eta}$$

Since $\beta q < 0$, $\lambda q > 0$, and $2\lambda > s\mu$, $\frac{\partial D^N}{\partial q} < 0$, so $\frac{\partial pN}{\partial q} > 0$.

Higher advertising effectiveness increases the profitability of advertising, which raises equilibrium advertising and, through the demand-enhancing effect of advertising, tends to increase the optimal price. The price-sensitivity channel (lower β) also contributes to higher prices through reduced consumer price sensitivity.

AI Effect on Equilibrium Advertising

Proposition 4 (AI and Advertising — Static Benchmark): Under the conditions of Proposition 3:

$$\frac{\partial a^N}{\partial q} = \frac{1}{\eta}\left[\lambda_q p^N + \lambda \frac{\partial p^N}{\partial q}\right] \quad > 0$$

Proof:

$$a^N = \frac{\lambda}{\eta} p^N, \quad \text{so} \quad \frac{\partial a^N}{\partial q} = \frac{1}{\eta}\left[\lambda_q p^N + \lambda \frac{\partial p^N}{\partial q}\right]$$

Since $\lambda q > 0$, $pN > 0$, and from Proposition 3, $\frac{\partial p^N}{\partial q} > 0$, the result follows.

Table 2: AI Effects Summary

Effect Channel	Static Benchmark Sign	Mechanism
$\frac{d\tilde{s}}{dq}$ (Reversal threshold)	+	$\frac{\kappa\lambda}{\mu} > 0$ (advertising channel)
$\frac{\partial p^N}{\partial q}$ (Price)	+	Both $\beta q < 0$ and $\lambda q > 0$ contribute
$\frac{\partial a^N}{\partial q}$ (Advertising)	+	Direct and indirect effects through p^N
AI expands reversal region	Yes	Through λq channel only

Welfare Analysis

Consumer Surplus

For linear demand with symmetric β, firm-level consumer surplus is:

$$CS_i = \frac{{D_i}^2}{2\beta}$$

At symmetry, market-wide consumer surplus is:

$$CS = CS_1 + CS_2 = \frac{D^2}{\beta}$$

Proposition 5 (Consumer Surplus Comparison): At a symmetric steady state with nonnegative equilibrium demand (D^C, $D^N \geq 0$):

$$CS^C - CS^N = \frac{(D^C)^2 - (D^N)^2}{\beta}$$

Therefore $CS^C > CS^N \iff D^C > D^N$, with:

$$D^C - D^N = -(\beta - s\gamma)(p^C - p^N) + \left(\frac{\alpha_1}{\delta} + \lambda - s\mu\right)(a^C - a^N)$$

Proof: From the demand equation at steady state, $G_i = a_i/\delta$. Since $p^C - p^N < 0$, the first term is positive. Since $a^C - a^N < 0$, the second term is negative. The net sign determines consumer surplus improvement.

Lower coordinated prices increase demand, but lower coordinated advertising decreases demand. The final welfare effect depends on which effect dominates.

Seller Welfare

Proposition 6 (Seller Welfare): Aggregate seller profit is:

$$\sum_i \Pi_i^C - \sum_i \Pi_i^N = 2\left((p^C - c)D^C - \frac{\eta}{2}(a^C)^2 - (p^N - c)D^N + \frac{\eta}{2}(a^N)^2\right)$$

Individual profit improvement ($\Pi_i^C > \Pi_i^N$ for both i) requires:

$$(p^C - c)D^C - \frac{\eta}{2}(a^C)^2 > (p^N - c)D^N - \frac{\eta}{2}(a^N)^2$$

Joint-profit maximization does not by itself imply an individual profit improvement; transfers or an explicit surplus-sharing mechanism may be required.

Steady-State Total Surplus

Proposition 7 (Steady-State Total Surplus): Let$W^{SS} = CS + \Pi_1 + \Pi_2$. Then:

$$W^{SS,C} - W^{SS,N} = \frac{{D^C}^2 - {D^N}^2}{\beta} + 2\left((p^C - c)D^C - \frac{\eta}{2}(a^C)^2 - (p^N - c)D^N + \frac{\eta}{2}(a^N)^2\right)$$

If aggregate consumer surplus and aggregate seller profit are both higher under coordination, then steady-state total surplus is higher. This is a steady-state surplus measure. Dynamic welfare would require integrating:

$$\int_0^\infty e^{-rt}W(t)dt$$

Which is beyond the scope of the current analysis. Advertising expenditure is treated as a real resource cost because it is subtracted from profits.

Table 3: Conditions for Consumer, Seller, and Total Surplus Improvement

Condition	Consumer Surplus	Seller Profit (Aggregate)	Total Surplus
$D^c > D^N$	+	Ambiguous	Ambiguous
$\Pi^c > \Pi^N$	Ambiguous	+	Ambiguous
Both conditions	+	+	+

Platform Policy Considerations

Platform Revenue Structure

The platform earns revenue from sales commission $\tau \in (0, 1)$ on each sale and from advertising fees from the sponsored auction:

$$\Pi^P = \tau \sum_i p_i D_i + R_A(a_1, a_2; R)$$

where $R_A(\cdot)$ denotes an advertising-auction revenue function to be specified in future numerical implementation.

Proposed Numerical Platform Extension

The platform extension is not solved in the present paper and is proposed as a numerical research design. The proposed numerical approach involves:

- Solving the seller equilibrium system for each (τ, R) grid point
- Computing platform revenue Π^P and steady-state surplus W^{ss}

- Identifying optimal τ^* and R^*
- Conducting sensitivity analysis

Table 4: Illustrative Parameterization

Parameter	Value	Justification
β_0	1.0	Normalization
γ	0.4	Illustrative
λ_0	0.5	Illustrative
μ	0.3	Illustrative
η	1.0	Normalization
δ	0.1	Illustrative
r	0.05	Illustrative
α_0	10.0	Illustrative
α_1	2.0	Illustrative
$\kappa\lambda$	0.5	Assumed
$\kappa\beta$	0.5	Assumed
q	1.0	Baseline
s	0.3	Baseline
c	0.0	Baseline

Illustrative Reversal Check: With $\lambda_0 = 0.5$, $\lambda(q) = 1.0$, the static reversal condition gives $0.3(1.0 - 0.09) = 0.273 < 0.4$, so no reversal occurs. With $\lambda_0 = 1.0$, $\lambda(q) = 1.5$, the condition gives $0.3(1.5 - 0.09) = 0.423 > 0.4$, so reversal occurs.

Policy Implications

Research on platform design suggests that commissions and reserve prices affect seller behavior through distinct channels, with different implications for platform revenue and consumer welfare (Hagiu & Wright, 2015).

1. Commission Effect

Commissions affect both pricing and advertising decisions through their impact on marginal profitability. An increase in the commission rate raises the effective marginal cost of each sale, leading sellers to increase prices. This price increase reduces consumer demand, which in turn reduces the marginal benefit of advertising. Consequently, higher commissions tend to reduce advertising intensity. The net effect on platform revenue is non-monotonic: while higher commissions generate more revenue per transaction, they also reduce transaction volume. An optimal commission rate balances these opposing forces.

2. Reserve Price Effect

In the proposed extension, reserve-price parameters would enter the advertising-auction component; their indirect effects on seller pricing would then depend on the specified auction and equilibrium structure. Reserve prices primarily affect the advertising side of seller decisions by establishing a minimum cost for sponsored placements. Higher reserve prices may exclude low-value advertisers, potentially reducing advertising competition and total advertising revenue. Unlike commissions, reserve prices do not directly affect the marginal profitability of sales, so their impact on pricing is indirect and depends on the strategic substitutability or complementarity between advertising and pricing decisions.

3. Policy Comparison

Commissions and reserve prices are not perfect substitutes as policy instruments. Commissions operate through the profit margin of every transaction, affecting both pricing and advertising incentives simultaneously. Reserve prices operate primarily through the advertising channel, leaving pricing decisions largely unaffected except through strategic interactions. This suggests that commission adjustments may generate larger welfare responses than reserve-price changes, although the magnitude depends on the specific parameter configuration.

4. Relation to Zhao and Berman (2025)

The proposed extension is motivated in part by the platform-policy analysis reported by Zhao and Berman (2025), although the present model does not reproduce their auction mechanism. Their study, using multi-agent reinforcement learning calibrated with Amazon data, found that reserve-price adjustments did not significantly increase platform profits, while commission adjustments had more pronounced effects. The analytical framework developed in this paper provides a structural interpretation of these findings: commissions affect both pricing and advertising incentives, whereas reserve prices affect only advertising, making commissions a more direct instrument for influencing equilibrium outcomes.

5. Calibration Caveat

In illustrative calibrations, commission adjustments may generate different welfare responses than reserve-price changes. This result is conditional on calibration and should not be interpreted as a general theorem. Future numerical implementation should explore the sensitivity of these policy effects across a wide range of parameter values, including different levels of search responsiveness, advertising effectiveness, and cross-price effects.

Conclusion

Summary of Findings

This paper has developed a differential game model of AI-driven dynamic pricing and advertising in e-commerce competition. Four main results are obtained.

First, in the static-goodwill benchmark, coordinated equilibrium prices fall below non-cooperative Nash equilibrium prices if and only if the advertising-spillover effect dominates the cross-price substitution effect:

$$p^C < p^N \iff \mu(\lambda - s\mu) > \eta\gamma$$

Second, the corresponding continuous-time differential game with endogenous goodwill is formulated and an affine-feedback solution class is investigated through the HJB coefficient system. Unlike the static benchmark, the dynamic condition is not closed-form because K^N and K^c depend on endogenous shadow values.

Third, in the static benchmark, higher AI capability expands the reversal region through

$$\frac{d\tilde{s}}{dq} = \frac{\kappa_\lambda}{\mu} > 0$$

and AI increases equilibrium prices and advertising through both price-sensitivity and advertising-effectiveness channels under the stated conditions.

Fourth, lower prices do not automatically guarantee higher consumer surplus when advertising and goodwill adjust; consumer surplus increases if and only if $D^C > D^N$ for nonnegative demand.

Contributions

To differential games, the Nerlove-Arrow (1962) framework has been extended to incorporate AI capability and search responsiveness, deriving precise conditions for coordination reversal in the static benchmark and formulating the dynamic extension.

To algorithmic competition, an analytical characterization of a mechanism that can generate the lower-price coordination outcome identified in recent computational research has been provided, deriving conditions that characterize when multidimensional coordination can generate lower prices.

To platform economics, the effects of platform policies on equilibrium outcomes have been outlined, providing a research design for future numerical analysis.

Limitations and Future Research

Several limitations suggest avenues for future research. The symmetric duopoly assumption could be relaxed to examine asymmetric firms, enhancing external validity. The linear demand specification could be replaced with nonlinear demand systems. Deterministic dynamics could be extended to incorporate stochastic shocks. The single-product assumption could be expanded to multi-product firms, capturing important strategic interactions. Empirical validation through direct structural estimation using platform data would strengthen the evidence base. Endogenous AI investment could be modeled to enrich the analysis. The existence and closure of the proposed quadratic-affine HJB solution class require formal coefficient verification and, where necessary, numerical solution.

Concluding Remarks

As e-commerce continues its trajectory toward AI-mediated competition, understanding the strategic interplay between pricing and advertising becomes increasingly critical. The differential game framework offers a rigorous foundation for this analysis, revealing both the risks and opportunities of algorithmic decision-making in digital markets.

The key insight that strategic coordination can generate lower prices under specific conditions suggests that algorithmic coordination may need to be evaluated across multiple strategic dimensions rather than through price outcomes alone. Regulators and platform operators should evaluate algorithmic coordination using both price and non-price dimensions, distinguishing potentially harmful coordination from configurations in which lower prices and reduced advertising expenditures can generate consumer benefits.

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Appendix A: Dynamic Hamilton-Jacobi-Bellman Coefficient System

A.1 Value-Function Derivatives

For the symmetric Nash case:

$$V^N(G_i, G_j) = v_0^N + v_1^N(G_i + G_j) + \frac{1}{2}v_3^N({G_i}^2 + {G_j}^2) + v_4^N G_i G_j$$

Therefore:

$$V_{Gi}^N = v_1^N + v_3^N G_i + v_4^N G_j, \quad V_{Gj}^N = v_1^N + v_4^N G_i + v_3^N G_j$$

For the cooperative case:

$$W^C(G_i, G_j) = w_0 + w_1(G_i + G_j) + \frac{1}{2}w_3({G_i}^2 + {G_j}^2) + w_4 G_i G_j$$

Therefore:

$$W_{Gi}^C = w_1 + w_3 G_i + w_4 G_j, \quad W_{Gj}^C = w_1 + w_4 G_i + w_3 G_j$$

A.2 Nash First-Order Conditions

From the price FOC $D_i - \beta\,(p_i - c) = 0$, expanding gives:

$$\alpha_0 + \alpha_1 G_i - \beta p_i + s\gamma p_j + \lambda a_i - s\mu a_j - \beta(p_i - c) = 0$$

$$2\beta p_i = \alpha_0 + \beta c + \alpha_1 G_i + s\gamma p_j + \lambda a_i - s\mu a_j$$

From the advertising FOC $\lambda\,(p_i - c) - \eta a_i + V_{Gi}^N = 0$:

$$\eta a_i = \lambda(p_i - c) + v_1^N + v_3^N G_i + v_4^N G_j$$

A.3 Nash Affine-Policy Coefficient Equations

Substitute $p_i = P_0^N + P_1^N G_i + P_2^N G_j$ and $a_i = A_0^N + A_1^N G_i + A_2^N G_j$ into the FOCs.

Constant terms:

$$2\beta P_0^N = \alpha_0 + \beta c + s\gamma P_0^N + \lambda A_0^N - s\mu A_0^N$$

$$\eta A_0^N = \lambda(P_0^N - c) + v_1^N$$

Linear in G_i:

$$2\beta P_1^N = \alpha_1 + s\gamma P_2^N + \lambda A_1^N - s\mu A_2^N$$

$$\eta A_1^N = \lambda P_1^N + v_3^N$$

Linear in G_j:

$$2\beta P_2^N = s\gamma P_1^N + \lambda A_2^N - s\mu A_1^N$$

$$\eta A_2^N = \lambda P_2^N + v_4^N$$

A.4 Nash Hamilton-Jacobi-Bellman Coefficient-Matching Equations

After substituting the affine policies into the Nash HJB:

$$rV_i^N = (p_i - c)D_i - \frac{\eta}{2}{a_i}^2 + V_{Gi}^N(a_i - \delta G_i) + V_{Gj}^N\big(a_j - \delta G_j\big)$$

and collecting powers of G_i, G_j, the coefficient equations are:

$$rv_3^N = Q_{ii}^N, \quad rv_4^N = Q_{ij}^N$$

where:

$$Q_{ii}^N = -\frac{\eta}{2}(A_1^N)^2 + A_1^N P_1^N \lambda - \beta(P_1^N)^2 + s\gamma P_1^N P_2^N + \alpha_1 P_1^N + v_3^N A_1^N + v_4^N A_2^N - \delta v_3^N$$

$$Q_{ij}^N = -\eta A_1^N A_2^N - s\mu A_1^N P_1^N + \lambda A_1^N P_2^N + \lambda A_2^N P_1^N - s\mu A_2^N P_2^N - 2\beta P_1^N P_2^N + s\gamma[\ (P_1^N)^2 + (P_2^N)^2] + v_3^N A_2^N + v_4^N A_1^N + v_4^N A_2^N + v_3^N A_1^N - 2\delta v_4^N$$

A.5 Cooperative First-Order Conditions

The full cooperative FOCs for firm i are:

Price:

$$D_i - \beta(p_i - c) + s\gamma\big(p_j - c\big) = 0$$

Advertising:

$$\lambda(p_i - c) - s\mu\big(p_j - c\big) - \eta a_i + W_{Gi}^C = 0$$

At a symmetric equilibrium, $p_i = p_j = p$ and $a_i = a_j = a$, so these conditions reduce to:

$$D^C = (\beta - s\gamma)(p^C - c), \quad a^C = \frac{(\lambda - s\mu)(p^C - c) + W_G^C}{\eta}$$

A.6 Cooperative Affine-Policy Coefficient Equations

Substitute $p_i = P_0^C + P_1^C G_i + P_2^C G_j$ and $a_i = A_0^C + A_1^C G_i + A_2^C G_j$ into the full cooperative FOCs.

Constant terms:

$$(\beta - s\gamma)P_0^C = \alpha_0 + (\beta - s\gamma)c + s\gamma P_0^C + \lambda A_0^C - s\mu A_0^C$$
$$\eta A_0^C = (\lambda - s\mu)\left(P_0^C - c\right) + w_1$$

Linear in G_i:

$$(\beta - s\gamma)P_1^C = \alpha_1 + s\gamma P_2^C + \lambda A_1^C - s\mu A_2^C$$
$$\eta A_1^C = (\lambda - s\mu)P_1^C + w_3$$

Linear in G_j:

$$(\beta - s\gamma)P_2^C = s\gamma P_1^C + \lambda A_2^C - s\mu A_1^C$$
$$\eta A_2^C = (\lambda - s\mu)P_2^C + w_4$$

A.7 Cooperative Hamilton-Jacobi-Bellman Coefficient-Matching Equations

After substituting the affine policies into the cooperative HJB and collecting powers of G_i, G_j, the coefficient equations are:

$$rw_3 = Q_{ii}^C, \quad rw_4 = Q_{ij}^C$$

where Q_{ii}^C and Q_{ij}^C are the cooperative analogues of the Nash expressions in A.4, with PkC, AkC, w1, w3, w4 replacing the Nash coefficients, and with the cooperative FOC structure reflected in the policy-coefficient equations.

A.8 Symmetric Steady-State Reduction

At steady state, $G_i = G_j = a/\delta$:

$$V_G^{N,ss} = v_1^N + (v_3^N + v_4^N)\frac{a^N}{\delta}$$

$$W_G^{C,ss} = w_1 + (w_3 + w_4)\frac{a^C}{\delta}$$

A.9 Admissibility and Verification Conditions

A candidate feedback equilibrium is admissible if the policy-coefficient system admits a solution, the value-function coefficient system admits a solution, the value functions are strictly concave with

$$v_3^N < 0, \quad |v_4^N| < -v_3^N, \quad w_3 < 0, \quad |w_4| < -w_3$$

The transversality condition holds:

$$\lim_{T\to\infty} e^{-rT} V\big(G(T)\big) = 0$$

controls are non-negative with p^*_i, $a^*_i \geq 0$, and

Denominators are positive with B^N, $B^c > 0$.

The equations above represent the HJB coefficient-matching system. Formal verification that this system closes within the proposed quadratic-affine class requires solving the coupled policy and value-function coefficient equations simultaneously. The system is presented as a candidate characterization; formal verification of existence and uniqueness is left for future work.

Summary of Key Mathematical Results

Table 5: Equilibrium Summary

Result	Equation	Status
Nash Price (Static)	$p^N = \dfrac{\alpha_0}{2\beta - s\gamma - \dfrac{\lambda(\lambda - s\mu)}{\eta}}$	Proven
Coordinated Price (Static)	$p^c = \dfrac{\alpha_0}{2(\beta - s\gamma) - \dfrac{\lambda(\lambda - s\mu)}{\eta}}$	Proven
Reversal Condition	$\mu(\lambda - s\mu) > \eta\gamma$	Proven
Reversal Region	$0 < s < \min\left\{1, \dfrac{\mu\lambda - \eta\gamma}{\mu^2}\right\}$	Proven
AI Effect on Threshold	$\dfrac{d\tilde{s}}{dq} = \dfrac{\kappa\lambda}{\mu} > 0$	Proven
AI Effect on Nash Price	$\dfrac{\partial p^N}{\partial q} = \dfrac{\alpha_0}{D_N^2}\left[-2\beta_q + \dfrac{(2\lambda - s\mu)\lambda q}{\eta}\right] > 0$	Proven
AI Effect on Advertising	$\dfrac{\partial a^N}{\partial q} = \dfrac{1}{\eta}\left[\lambda_q p^N + \lambda\dfrac{\partial p^N}{\partial_q}\right] > 0$	Proven
Consumer Surplus	$CS^c > CS^N \Leftrightarrow D^c > D^N$ (for $D^c, D^N \geq 0$)	Proven
Demand Difference	$D^c - D^N = -(\beta - s\gamma)(p^c - p^N) + \Lambda(a^c - a^N)$	Proven
Dynamic Proposition 1	$A^cB^N < A^NB^c$	Conditional
Cooperative SOC	$2\eta(\beta - s\gamma) > (\lambda - s\mu)^2$ and $2\eta(\beta + s\gamma) > (\lambda + s\mu)^2$	Corrected

Table 6: Notational Summary

Symbol	Meaning	Range
s	Search responsiveness	(0, 1)
q	AI capability	$[0, \infty)$
$\beta(q)$	Price sensitivity	$\beta_0 / (1 + \kappa\beta q)$
$\lambda(q)$	Advertising effectiveness	$\lambda_0 + \kappa\lambda q$
γ	Cross-price effect	$(0, \beta)$
μ	Cross-advertising effect	> 0
η	Advertising cost convexity	> 0
δ	Goodwill depreciation	> 0
r	Discount rate	> 0
τ	Platform commission	(0, 1)
R	Reserve price	$[0, \infty)$